Prehistory problem for systems driven by white noise

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The prehistory probability density for a stochastic system with Gaussian white noise is analyzed in terms of a Fokker-Planck equation. The results are compared with those provided by the method of the optimal path, and with those of analog simulation experiments. [S1063-651X(96)07607-6]

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In recent years, new approaches to the investigation of rare events in systems driven by noise have been proposed. In [1,2], the description of large rare excursions of the dynamical variable away from the stable point is given in terms of a new statistical quantity, the so called *prehistory probability density*. The prehistory problem can be briefly formulated as follows. Let us assume a system at equilibrium so that the one-time probability density $P_{\rm st}(x)$ does not depend on time. If we know that at time t_f the system is at x_f, we ask for the probability density $p_h(x,t;x_f,t_f)$ that the system was at point x at time $t<t_f$. So we want to know about the behavior of the system prior to the fulfillment of a given condition. Dykman *et al.* formulate the problem in terms of a variational principle which allows them to express the prehistory distribution in terms of a Gaussian centered about the optimal path in the limit of small noise strength. For bistable potentials, the dispersion might show a nonmonotonic behavior which is more pronounced the further the point x_f is from a stable position. Also in [1] the results of analog experiments are compared with the theoretical ideas. Although the theory provides a good qualitative description of the experiments, there are some quantitative discrepancies in the behavior of the dispersion especially for large deviations from the stable point. In this report, we present the results obtained using an alternative formulation of the prehistory problem which is adequate for white noise, and does not require the use of a variational principle.

Let us consider a system described by a single stochastic variable, whose dynamics is given by the Langevin equation

$$\dot{x}(t)=-U'(x)+\xi(t), \tag{1}$$

where $\xi(t)$ is a Gaussian white noise with

$$\langle \xi(t)\rangle=0, \quad \langle \xi(t)\xi(s)\rangle=D\,\delta(t-s). \tag{2}$$

Let $w_2(x_i,t_i;x_f,t_f)$ and $w_3(x_i,t_i;x,t;x_f,t_f)$ be the two- and three-time joint probability densities for times $t_i<t<t_f$, of the stochastic process $x(t)$. Following [1], we define the prehistory probability density as

$$p_h(x,t;x_f,t_f)=\left(\frac{w_3(x_i,t_i;x,t;x_f,t_f)}{w_2(x_i,t_i;x_f,t_f)}\right)_{t_i\to-\infty}, \tag{3}$$

where the particularization for $t_i\to-\infty$ indicates that the system was prepared in some state x_i in the far past and then, at any finite time like t or t_f, we will find it at equilibrium.

Using the definitions of w_2 and w_3 in terms of conditional probability densities, we can immediately write

$$p_h(x,t;x_f,t_f)=\frac{P_{\rm st}(x)}{P_{\rm st}(x_f)}p_{1|1}(x,t|x_f,t_f), \tag{4}$$

where $p_{1|1}(x,t|x_f,t_f)$ $(t<t_f)$ is the solution of the backward Fokker-Planck equation with the condition $p_{1|1}(x_f,0|x,0)=\delta(x-x_f)$. If we use now the symmetry property of the two-times joint probability density with respect to the exchange of arguments [3] and particularize it for a stationary Markovian process, we get

$$p_{1|1}(x,t|x_f,t_f)\frac{P_{\rm st}(x)}{P_{\rm st}(x_f)}\quad (t<t_f)=p_{1|1}(x,t|x_f,t_f)\quad (t>t_f). \tag{5}$$

The conditional probability density on the right hand side of Eq. (5) can be identified as a function of its first argument, with the *forward* probability density $p^f_{1|1}(x,t|x_f,t_f)(t>t_f)$, whose temporal evolution is given by the solution of the forward Fokker-Planck equation

$$\frac{\partial}{\partial t}p^f_{1|1}(x,t|x_f,t_f)=\mathcal{D}(x)p^f_{1|1}(x,t|x_f,t_f), \tag{6}$$

with the initial condition $p^f_{1|1}(x,0|x_f,0)=\delta(x-x_f)$, and where

$$\mathcal{D}(x)=\frac{\partial}{\partial x}U'(x)+\frac{D}{2}\frac{\partial^2}{\partial x^2}. \tag{7}$$

Combining Eq. (4) and Eq. (5), we can then write

$$p_h(x,t;x_f,t_f)\quad (t<t_f)=p^f_{1|1}(x,t|x_f,t_f)\quad (t>t_f), \tag{8}$$

which shows that the prehistory distribution for times t prior to the final observation time $t=t_f$ can be found by solving the appropriate forward Fokker-Planck (FP) problem.

The Fokker-Planck equation cannot, in general, be solved in an exact way, so we must resort to numerical integration schemes. A useful method for the study of the time dependent solutions of the forward Fokker-Planck equation is based on the split operator scheme introduced by Feit *et al.* to solve the time dependent Schrödinger equation [4]. We have applied this method to problems of the type given by Eq. (6) with different types of drift terms [5]. As we are interested in the knowledge of the prehistory probability den-

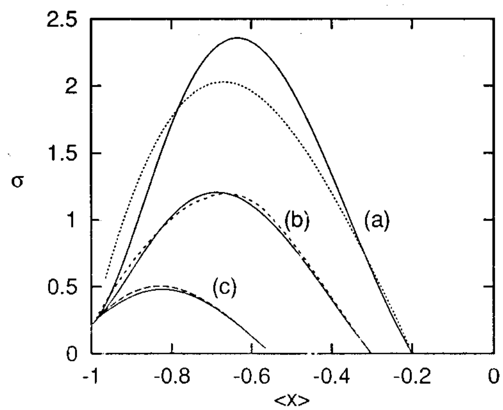

FIG. 1. Plot of the dispersion parameter $\sigma(\langle x\rangle,x_f)$ for a bistable potential with noise strength $D=0.027$ and $x_f=-0.2$ (case a), $x_f=-0.3$ (case b), and $x_f=-0.55$ (case c). Solid lines correspond to the results of Eq. (10); broken lines correspond to our numerical results.

sity we have solved the forward FP equation with an initial condition given by a δ function centered at different values of x_f. We have considered potentials of the form

$$U(x)=a\frac{x^2}{2}+b\frac{x^4}{4}, \tag{9}$$

and so, for different values of the coefficients a and b, we can analyze the dynamics for both mono- and bistable situations.

In Fig. 1 we have considered the case of a bistable potential with $a=-1$ and $b=1$, and different values of x_f. We have taken $D=0.027$, which is small enough to allow us to compare with the asymptotic limit of Dykman *et al.* and, at the same time, ensures the convergence of the numerical procedure. The FP equation is solved during time intervals such that the probability density remains localized in the original well. From the solution, it is straightforward to compute the first two cumulants, and from here, the noise independent dispersion parameter $\sigma(\langle x\rangle,x_f)=\langle x^2\rangle-\langle x\rangle^2/D$,

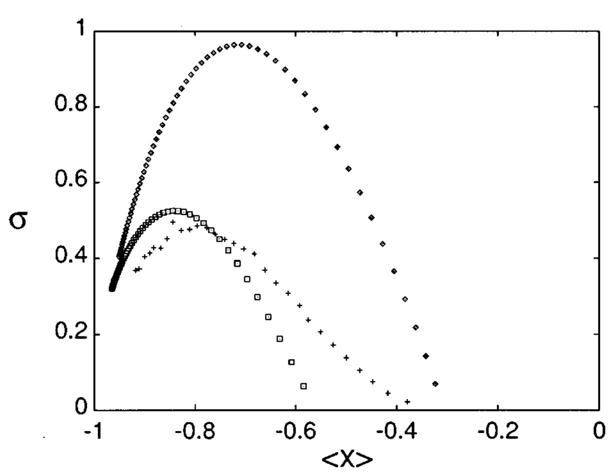

FIG. 2. Plot of the numerical results for $\sigma(\langle x\rangle,x_f)$ for a bistable potential with noise strength $D=0.07$ and $x_f=-0.3$ (diamonds), $x_f=-0.565$ (squares). The plus signs correspond to the experimental values of Dykman *et al.* for the same noise strength.

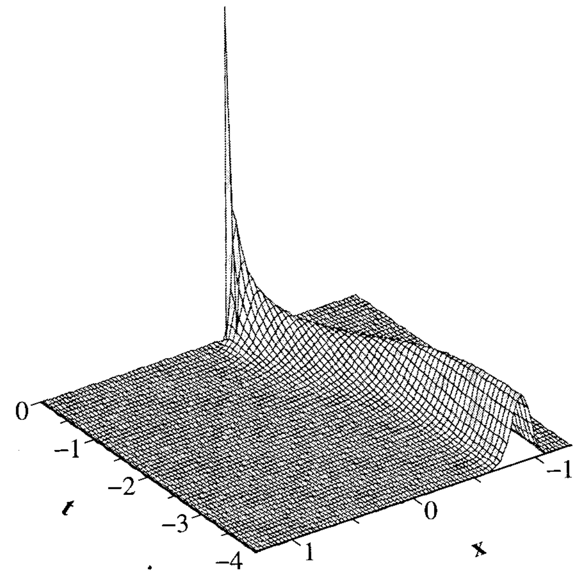

FIG. 3. The prehistory probability density for a bistable potential with $D=0.027$ and $x_f=-0.3$, as a function of time.

evaluated as a function of $\langle x\rangle$ for each value of x_f. In the Gaussian approximation of [1], this quantity is given by

$$\sigma(x,x_f)=[U'(x)]^2\int_x^{x_f}dy[U'(y)]^{-3}, \tag{10}$$

where $x=x_{\rm op}$, $x_{\rm op}$ being the optimum value of the stochastic variable at each instant of time $t<0$, obtained from the solution of

$$\dot{x}_{\rm op}=U'(x_{\rm op})$$

with the condition $x_{\rm op}(0)=x_f$. It is clear from the plots that the results of the numerical solution match quite well those of the analytical theory. In particular, we notice the non-

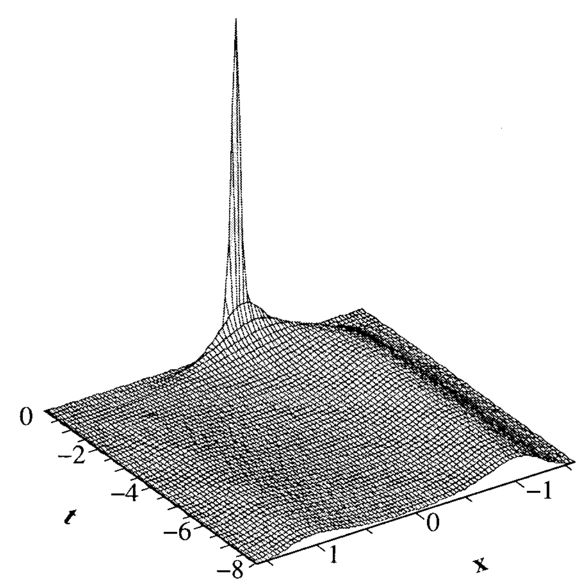

FIG. 4. The prehistory probability density for a bistable potential with $D=0.3$ and $x_f=-0.3$, as a function of time.

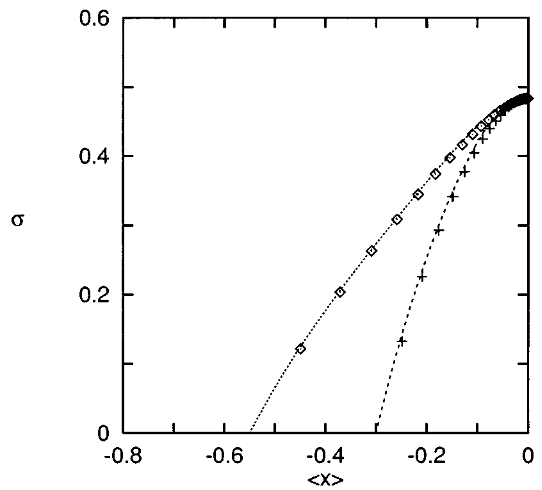

FIG. 5. Plot of the dispersion parameter $\sigma(\langle x\rangle, x_f)$ for a monostable potential with noise strength $D=0.027$ and $x_f=-0.3,-0.55$. Broken lines correspond to the results obtained with Eq. (10); the symbols correspond to our numerical results.

monotonic behavior of the dispersion, which is more pronounced the farther x_f is from the stable point. The first moment, as well as the optimum path, does not show the sudden drop at $t=0$ observed in analog simulations [1]. We have also checked that the dispersion curves are quite independent of the noise strength, as long as it is small enough to guarantee that the probability distribution does not develop a second peak during the time interval of interest. Thus the Gaussian approximation around the optimal path provides a good description even if D is not asymptotically small. Clearly, as the observation point x_f gets closer to the unstable point, the values of D needed to guarantee this independence get smaller.

Our numerical scheme remains valid for noise strengths that are not too small, and it is certainly adequate for the value of D quoted in Fig. 1(a) of [1]. Thus it seems worthwhile to investigate the dispersion behavior for this noise strength to analyze the discrepancy between analytical predictions and experimental findings. In Fig. 2, we show the output of our calculation for $x_f=-0.3$ and $D=0.07$. It is clear that the dispersion values deviate largely from those observed in the experiments. On the other hand, if we keep the same noise strength, but we take the end point to be $x_f=-0.3-D^{1/2}=-0.565$, the numerical and experimental curves agree well for values of x up to $x\approx-0.7$. These results indicate that the discrepancy between theory and experiments is probably due to the fact that the final part of a trajectory seen in the analog experiments consists of a sudden jump of order $D^{1/2}$ [6]. If we take into account this uncertainty in the experimental determination of large x, the coincidence between calculated and experimental values of the dispersion for these large excursions improves.

In Fig. 3, we show the time evolution of the probability density for the parameters $D=0.027$ and $x_f=-0.3$. Notice that at intermediate times the distribution is broader and its peak is lower than at longer times, when a kind of focusing effect is noticeable. During the time interval shown in this figure, the distribution remains single peaked. Deviations from Gaussian character are not too strong, as indicated by the very small values of the cumulants of order higher than 2. The effects on the dynamics as noise is increased can be seen in Fig. 4, where we show the probability density for a higher value of the noise strength, $D=0.3$, and $x_f=-0.3$. It is clear that now the noise is so large that transitions between wells are relevant even for not very long times and thus the distribution quickly develops a second maximum. Thus, from the point of view of the stochastic trajectories, this means that there are also contributions to the prehistory probability density from trajectories which have been able to overcome the barrier before arriving at x_f at the observation time. During this time regime, the Gaussian approximation around a single optimum path would fail.

In Fig. 5, we show the behavior of the dispersion for the monostable case, $a=1, b=1$, corresponding to $D=0.027$ for two different values of x_f. The analytical results obtained with Eq. (10) agree very well with our numerical solution. The nonmonotonic behavior typical of bistable cases does not show up here. This feature indicates that the nonmonotony is due to the fact that the potential energy curve has more than one minimum, rather than to the anharmonicity of the potential. Also, deviations from the Gaussian character of the probability density are much smaller than in the bistable situation.

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